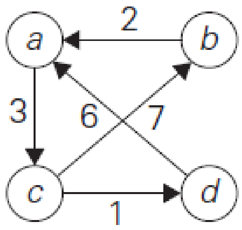
FLOYD'S ALGORITHM

- The A***ll-pairs Shortest Paths Problem*** finds the distances—i.e., the lengths of the shortest paths— from each vertex to all other vertices.
- ***Floyd's algorithm*** invented by Robert W. Floyd. is used to solve All-pairs shortest paths problem.
- It is applicable to both undirected and directed weighted graphs.



(a)

$$W = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \infty & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \infty & 0 \end{bmatrix} \end{array}$$

(b)

$$D = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & 10 & 3 & 4 \\ 2 & 0 & 5 & 6 \\ 7 & 7 & 0 & 1 \\ 6 & 16 & 9 & 0 \end{bmatrix} \end{array}$$

(c)

(a) Digraph. (b) Its weight matrix. (c) Its distance matrix.

- Floyd's algorithm computes the distance matrix of a weighted graph with n vertices through a series of $n \times n$ matrices:

$$D^{(0)}, \ldots, D^{(k-1)}, D^{(k)}, \ldots D^{(n)}$$

- The element $d^{(k)}_{ij}$ in the ith row and the jth column of matrix $D^{(k)}$ $(i, j = 1, 2, \ldots, n, k = 0, 1, \ldots, n)$ is equal to the length of the shortest path among all paths from the ith vertex to the jth vertex with each intermediate vertex, if any, numbered not higher than k.
- $D^{(0)}$ is simply the weight matrix of the graph. The last matrix in the series, $D^{(n)}$, contains the lengths of the shortest paths among all paths that can use all n vertices as intermediate.
- The formula for generating the elements of matrix $D^{(k)}$ from the elements of matrix $D^{(k-1)}$:

$$d_{ij}^{(k)} = \min\{d_{ij}^{(k-1)},\ d_{ik}^{(k-1)} + d_{kj}^{(k-1)}\} \quad \text{for } k \geq 1, \quad d_{ij}^{(0)} = w_{ij}.$$

- That is, the element in row i and column j of the current distance matrix $D(k-1)$ is replaced by the sum of the elements in the same row i and the column k and in the same column j and the row k if and only if the latter sum is smaller than its current value.
- The pseudocode of Floyd's algorithm is

  ALGORITHM *Floyd(W[1..n, 1..n])*

  //Implements Floyd's algorithm for the all-pairs shortest-paths problem

  //Input: The weight matrix W of a graph with no negative-length cycle

//Output: The distance matrix of the shortest paths' lengths

D ← *W*

for *k*←1 **to** *n* **do**

for *i* ←1 **to** *n* **do**

for *j* ←1 **to** *n* **do**

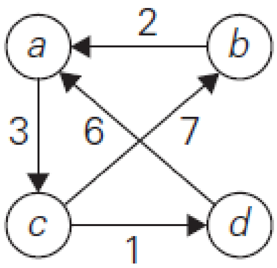
D[i, j] ← min{D[*i*, *j*], D[*i*, *k*] + D[*k*, *j*]*)*

return D

- The time efficiency is only $\Theta(n^3)$.

PROBLEM

Solve the ***all-pairs shortest paths problem*** for the given graph



The weight matrix for the given graph is

$$D^{(0)} = \begin{array}{c|cccc} & a & b & c & d \\ \hline a & 0 & \infty & 3 & \infty \\ b & 2 & 0 & \infty & \infty \\ c & \infty & 7 & 0 & 1 \\ d & 6 & \infty & \infty & 0 \end{array}$$

To find $D^{(1)}$, i.e. Lengths of the shortest paths with intermediate vertices numbered not higher than 1, i.e., just *a*

$D^{(1)}[b,c] = \min\{D^{(0)}[b,c], D^{(0)}[b,a] + D^{(0)}[a,c]\} = \min\{\infty, 2 + 3\} = \mathbf{5}$

$D^{(1)}[b,d] = \min\{D^{(0)}[b,d], D^{(0)}[b,a] + D^{(0)}[a,d]\} = \min\{\infty, 2 + \infty\} = \infty$

$D^{(1)}[c,b] = \min\{D^{(0)}[c,b], D^{(0)}[c,a] + D^{(0)}[a,b]\} = \min\{7, \infty + \infty\} = 7$

$D^{(1)}[c,d] = \min\{D^{(0)}[c,d], D^{(0)}[c,a] + D^{(0)}[a,d]\} = \min\{1, \infty + \infty\} = 1$

$D^{(1)}[d,b] = \min\{D^{(0)}[d,b], D^{(0)}[d,a] + D^{(0)}[a,b]\} = \min\{\infty, 6 + \infty\} = \infty$

$D^{(1)}[d,c] = \min\{D^{(0)}[d,c], D^{(0)}[d,a] + D^{(0)}[a,c]\} = \min\{\infty, 6 + 3\} = \mathbf{9}$

Now our $D^{(1)}$ is

$$D^{(1)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & \mathbf{5} & \infty \\ \infty & 7 & 0 & 1 \\ 6 & \infty & \mathbf{9} & 0 \end{bmatrix} \end{array}$$

To find $D^{(2)}$, i.e. Lengths of the shortest paths with intermediate vertices numbered not higher than 2, i.e., *a & b*

$D^{(2)}[a,c] = \min\{D^{(1)}[a,c], D^{(1)}[a,b] + D^{(1)}[b,c]\} = \min\{3, \infty + 5\} = 3$

$D^{(2)}[a,d] = \min\{D^{(1)}[a,d], D^{(1)}[a,b] + D^{(1)}[b,d]\} = \min\{\infty, \infty + \infty\} = \infty$

$D^{(2)}[c,a] = \min\{D^{(1)}[c,a], D^{(1)}[c,b] + D^{(1)}[b,a]\} = \min\{\infty, 7 + 2\} = \mathbf{9}$

$D^{(2)}[c,d] = \min\{D^{(1)}[c,d], D^{(1)}[c,b] + D^{(1)}[b,d]\} = \min\{1, 7 + \infty\} = 1$

$D^{(2)}[d,a] = \min\{D^{(1)}[d,a], D^{(1)}[d,b] + D^{(1)}[b,a]\} = \min\{6, \infty + 2\} = 6$

$D^{(2)}[d,c] = \min\{D^{(1)}[d,c], D^{(1)}[d,b] + D^{(1)}[b,c]\} = \min\{9, \infty + 5\} = 9$

Now our $D^{(2)}$ is

$$D^{(2)} = \begin{array}{c} \\ a \\ b \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccc} a & b & c & d \end{array} \\ \begin{bmatrix} 0 & \infty & 3 & \infty \\ 2 & 0 & 5 & \infty \\ \mathbf{9} & 7 & 0 & 1 \\ 6 & \infty & 9 & 0 \end{bmatrix} \end{array}$$

To find $D^{(3)}$, i.e. Lengths of the shortest paths with intermediate vertices numbered not higher than 3, i.e., *a, b & c*

$D^{(3)}[a,b] = \min\{D^{(2)}[a,b], D^{(2)}[a,c] + D^{(2)}[c,b]\} = \min\{\infty, 3 + 7\} = \mathbf{10}$

$D^{(3)}[a,d] = \min\{D^{(2)}[a,d], D^{(2)}[a,c] + D^{(2)}[c,d]\} = \min\{\infty, 3 + 1\} = \mathbf{4}$

$D^{(3)}[b,a] = \min\{D^{(2)}[b,a], D^{(2)}[b,c] + D^{(2)}[c,a]\} = \min\{2, 5 + 9\} = 2$

$D^{(3)}[b,d] = \min\{D^{(2)}[b,d], D^{(2)}[b,c] + D^{(2)}[c,d]\} = \min\{\infty, 5 + 1\} = \mathbf{6}$

$D^{(3)}[d,a] = \min\{D^{(2)}[d,a], D^{(2)}[d,c] + D^{(2)}[c,a]\} = \min\{6, 9 + 9\} = 6$

$D^{(3)}[d,b] = \min\{D^{(2)}[d,b], D^{(2)}[d,c] + D^{(2)}[c,b]\} = \min\{\infty, 9 + 7\} = \mathbf{16}$

Now our $D^{(3)}$ is

$$D^{(3)} = \begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & \mathbf{10} & 3 & \mathbf{4} \\ b & 2 & 0 & 5 & \mathbf{6} \\ c & 9 & 7 & 0 & 1 \\ d & 6 & \mathbf{16} & 9 & 0 \end{array}$$

To find $D^{(4)}$, i.e. Lengths of the shortest paths with intermediate vertices numbered not higher than 4, i.e., *a, b, c & d*

$D^{(4)}[a,b] = \min\{D^{(3)}[a,b], D^{(3)}[a,d] + D^{(3)}[d,b]\} = \min\{10, 4 + 16\} = 10$

$D^{(4)}[a,c] = \min\{D^{(3)}[a,c], D^{(3)}[a,d] + D^{(3)}[d,c]\} = \min\{3, 4 + 9\} = 3$

$D^{(4)}[b,a] = \min\{D^{(3)}[b,a], D^{(3)}[b,d] + D^{(3)}[d,a]\} = \min\{2, 6 + 6\} = 2$

$D^{(4)}[b,c] = \min\{D^{(3)}[b,c], D^{(3)}[b,d] + D^{(3)}[d,c]\} = \min\{5, 6 + 9\} = 5$

$D^{(4)}[c,a] = \min\{D^{(3)}[c,a], D^{(3)}[c,d] + D^{(3)}[d,a]\} = \min\{9, 1 + 6\} = \mathbf{7}$

$D^{(4)}[c,b] = \min\{D^{(3)}[c,b], D^{(3)}[c,d] + D^{(3)}[d,b]\} = \min\{7, 1 + 16\} = \mathbf{7}$

Now our $D^{(4)}$ is

$$D^{(4)} = \begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & 10 & 3 & 4 \\ b & 2 & 0 & 5 & 6 \\ c & \mathbf{7} & 7 & 0 & 1 \\ d & 6 & 16 & 9 & 0 \end{array}$$

The shortest [ath from every vertex to every other vertex present in the given graph is

$$\begin{array}{c|cccc|} & a & b & c & d \\ a & 0 & 10 & 3 & 4 \\ b & 2 & 0 & 5 & 6 \\ c & \mathbf{7} & 7 & 0 & 1 \\ d & 6 & 16 & 9 & 0 \end{array}$$